Constrained optimization with second order conditions

Find whether the conditional extremes of the following function exist:

$$f(x,y) = 3x^2 + y^2 - x + 1 \text{ Subject to: } x^2 + \frac{y^2}{4} = 1$$

Solution

We build the Lagrangian:

$$L = 3x^2 + y^2 - x + 1 + \lambda(1 - x^2 - y^2/4)$$

$$L'x = 6x - 1 - 2x\lambda = 0$$

$$L'y = 2y - \lambda y/2 = 0$$

$$L'\lambda = 1 - x^2 - y^2/4 = 0$$

From the first two conditions.

$$x(6 - 2\lambda) = 1$$

$$4y = \lambda y$$

The second equation can be satisfied if $y = 0$. We use this in the third equation:

$$1 - x^2 - 0 = 0$$

$$x^2 = 1$$

Which is satisfied with $x = 1$ and $x = -1$. Therefore we have 2 possible critical points: $(1, 0)$, $(-1, 0)$. Now let's go for the other case where we assume that $y \neq 0$ and we can work with the second equation to find the value of λ:

$$4 = \lambda$$

This gives us the following value of x

$$6x - 1 - 2x4 = 0$$

$$-2x = 1$$

$$x = -1/2$$

Going to the third equation:

$$1 - (-1/2)^2 - y^2/4 = 0$$

$$1 - 1/4 - y^2/4 = 0$$

$$3 = y^2$$

Which tells us that $y = \sqrt{3}$ or $y = -\sqrt{3}$. With this we have two more possible extremes: $(-1/2, \sqrt{3})$ and $(-1/2, -\sqrt{3})$. We calculate the second derivatives to evaluate the second order condition:

$$f''_{xx} = 6 - 2\lambda$$

$$f''_{yy} = 2 - \lambda/2$$

$$f''_{xy} = f''_{yx} = 0$$

We also have the derivatives of the constraint:

$$g'x = 2x$$

$$g'y = y/2$$

This generates the following bordered Hessian:

$$\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''_{xx} & L''_{xy} \\ g'y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 2x & y/2 \\ 2x & 6 - 2\lambda & 0 \\ y/2 & 0 & 2 - \lambda/2 \end{pmatrix}$$

We calculate the determinant:

$$-2x \begin{vmatrix} 2x & y/2 \\ 0 & 2 - \lambda/2 \end{vmatrix} + y/2 \begin{vmatrix} 2x & y/2 \\ 6 - 2\lambda & 0 \end{vmatrix}$$

$$-2x[2x(2-\lambda/2)] + y/2[-(6-2\lambda)y/2] = -2x[4x - x\lambda] + y/2[-3y + y\lambda] = -8x^2 + 2x^2\lambda - y^2\frac{3}{2} + y^2\lambda/2$$

Simplifying:

$$|\bar{H}| = x^2(-8+2\lambda) + \frac{y^2}{2}(-3+\lambda)$$

This we have to evaluate at the following points (and calculating the value of λ for each one):

- $(1,0)$ and $\lambda = 5/2$
- $(-1,0)$ and $\lambda = 7/2$
- $(-1/2, \sqrt{3})$ and $\lambda = 4$
- $(-1/2, -\sqrt{3})$ and $\lambda = 4$

We evaluate the 4 cases:

- $$-8+5 = -3 < 0$$

  We are in front of a minimum.

- $$-8+7 = -1 < 0$$

  We are in front of a minimum.

- $$\frac{1}{16}(-8+2*4) + \frac{3}{2}(-3+4) = 3/2 > 0$$

  We are in front of a maximum.

- $$\frac{1}{16}(-8+2*4) + \frac{3}{2}(-3+4) = 3/2 > 0$$

  We are in front of a maximum.